INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 27, 169–177 (1998)

ROUGH BOUNDARIES AND WALL LAWS

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SUMMARY

We describe a new approach for developing new wall-laws for rough surfaces. We also give error estimates on a simple model. © 1998 John Wiley & Sons, Ltd.

Int. J. Numer. Meth. Fluids, **27**: 169–177 (1998)

KEY WORDS: wall laws; wavy surfaces; turbulence

1. INTRODUCTION

In a number of important applications the domain occupied by the fluid has two scales. We shall consider two such cases: 1, the flow over a rough surface; 2, the flow over a wavy sea surface. In case 1 it could be a badly polished flat plate or a surface with periodic ridges such as the tiles of a re-entry vehicle or the effect of trees and buildings on a meteorological flow.

It is not possible to discretize numerically both scales with sufficient accuracy; a boundary condition is sought which takes into account the roughness of the surface. The usual answer is given by the law of the wall with a different β:

$$u^+ = \frac{1}{\chi}\log y^+ + \beta$$

on a mean surface Σ above the wavy surface. This formula is used to establish a non-linear Frechet boundary condition

$$u\cdot n = 0, \qquad \frac{u\cdot s}{\sqrt{(\nu_T|\partial u/\partial n|)}} - \frac{1}{\chi}\log\left[\delta\sqrt{\left(\frac{1}{\nu_T}\left|\frac{\partial u}{\partial n}\right|\right)}\right] + \beta = 0,$$

where ν_T is the turbulent viscosity. Here we wish to show that it could also be derived from another generalized Frechet condition

$$\nu_T(\nabla u + \nabla u^T)n - pn = \sigma n_\Sigma = c(|u|)u$$

which comes from domain decomposition and has nothing to do with wall laws.

In case 2 the surface drag of the wind on the wavy sea must be taken into account in meteorological

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CCC 0271–2091/98/010169–09 $17.50

Received May 1996

air models and *vice versa*. A boundary condition of the type

$$\sigma_s n_\Sigma = \sigma_a n_\Sigma = c|u_a - u_s|(u_s - u_s)$$

is proposed. Here again we want to show how domain decomposition can explain this condition.
This work is an extension of that of Carreau,[1] Le Tallec[2] and Achdou *et al.*[3,9]

2. ROUGH TERRAIN

Consider the Reynolds-averaged Navier–Stokes equations with a turbulence model for ν_T:

$$u\nabla u + \nabla p - \nabla\cdot[\nu_T(\nabla u + \nabla u^T)] = 0, \qquad \nabla\cdot u = 0$$

on a domain Ω^ε.

Let us seek a solution by domain decomposition[2] (Figure 1). Let $\Sigma//\Gamma$ and $\Omega = \Omega_o \cup \Omega_i$. Let u_i be the solution in Ω_i with $u = v$ on Σ. Let u_o be the solution in Ω_o with $u = v$ on Σ. We have a solution to the problem if v is such that normal stresses match:

$$\sigma_i \cdot n = \sigma_o \cdot n.$$

Now by definition of u_i we know that the solution is a function of v, so its normal stress on the upper wall is also a function of v:

$$\sigma_i \cdot n = F(v).$$

Thus the continuity of σ gives

$$\sigma_o \cdot n = F(u_o).$$

The trouble, however, is that F is in general a non-local operator.

2.1. Periodic irregularities

For periodic irregularities F becomes approximatively local, because the solution u_i can be found by translation of the solution u' of a single-cell problem with only one irregularity at the lower boundary, periodic conditions on the vertical boundaries and matching condition $u' = v$ at the top boundary. Then this cell problem is solved for all values of v and a table is made of $\nu_T(\nabla u_i + \nabla u_i^T)n - p_i n)|_\Sigma$ versus v. See Figure 2 for an example of a solution on such surfaces.

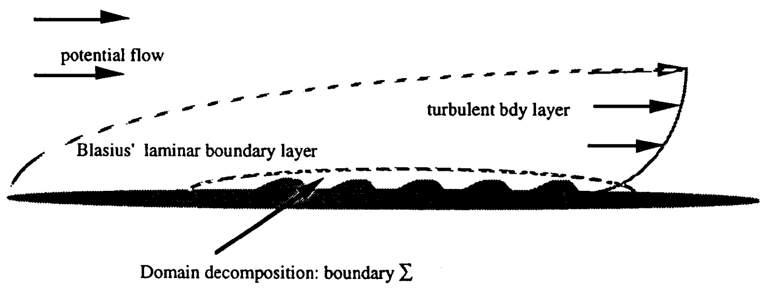

Figure 1. Domain decomposition for flow over a rough surface

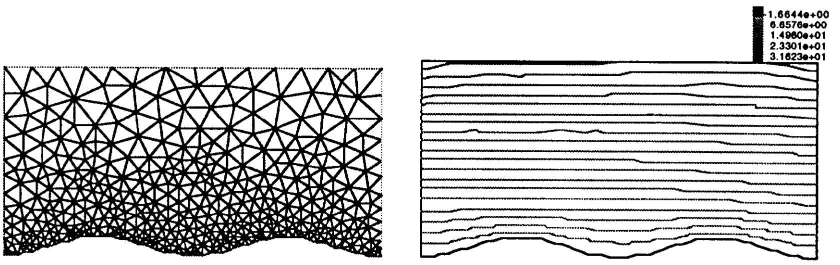

Figure 2. 2D Navier–Stokes equation at $Re = \nu^{-1} = 50^4$

2.2. *Remark*

Notice that by the divergence theorem and Green's theorem,

$$\int_{\partial\Omega_i} [\nu_T(\nabla u + \nabla u^T)n - pn] = \int_{\Omega_i} \{\nabla\cdot[\nu_T)\nabla u + \nabla u^T)] - \nabla p\} = \int_{\Omega_i} \nabla\cdot(u \otimes u) = \int_{\partial\Omega_i} uu\cdot n = 0.$$

Therefore

$$-\int_{\partial\Omega\cap\Omega_i} [\nu_T(\nabla u + \nabla u^T)n - pn] = \int_\Sigma [\nu_T(\nabla u + \nabla u^T)n - pn].$$

Thus F(u) is also the drag of the rough surface per unit length. This means that tabulations of F could also be done experimentally.

3. OCEAN–ATMOSPHERE INTERFACE

Consider now a large domain Ω with air above and water below an interface S. The Reynolds-averaged Navier–Stokes equations for Ω contain a variable density $\rho(x, t)$ and a turbulent viscosity μ_T which changes with the medium:

$$\rho(\partial_t u + u\nabla u) + \nabla p - \nabla\cdot[\mu_T(\nabla u + \nabla u^T)] = 0,$$
$$\partial_t\rho + u\nabla\rho = 0, \qquad \nabla\cdot u = 0.$$

Let us split Ω into three parts (Figure 3): 1, a thin domain Ω_i containing the interface; 2, the remainder of Ω in water, Ω_s; 3, the remainder of Ω in air, Ω_a. Σ_s and Σ_a are the boundaries between Ω_i and Ω_s, Ω_a respectively and u_a and u_s are the velocities on these interfaces.

The solution in an infinite horizontal slab Ω_i with $u = u_a$ on top and $u = u_s$ on the bottom will be close to the exact solution, because the effects of upstream and downstream errors decay exponentially with the distance to these. Frame invariance can be utilized to put to zero one of the

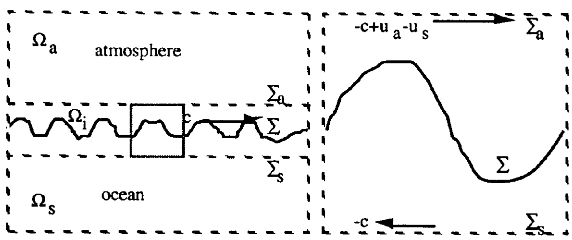

Figure 3. Domain decomposition for free surface flow

non-homogeneous boundary conditions: in the slab this changes $\{u_s, u_a\}$ into $\{0, u_a - u_s\}$. It shows that the tabulation is to be a function of one parameter $u_a - u_s$ and not the two independant parameters $\{u_s, u_a\}$. Now use again frame invariance to make the problem stationary in time. If all surface waves travel at the same speed c, then use $\{-c, u_a - u_s - c\}$ as top and bottom boundary conditions. Finally approximate the infinite slab by periodic translations of a finite representative cell C and tabulate

$$\sigma_n|_{\Sigma_a} = \sigma_n|_{\Sigma_s} = F(c, u_a - u_s).$$

3.1. Remark on tabulations for waves

Consider the problem of finding F:

$$\mu_T(\nabla u + \nabla u^T)n - pn = F([u]) \quad \text{on } \Sigma,$$

where u is the solution of the Navier–Stokes equations with a turbulence model and varying density and [.] denotes the jump across the interface. Integration by parts shows that the mean values on Σ_a and Σ_o are equal to the mean jump of the same at the interface, which is also the drag of the wavy free surface. In many engineering applications it is considered reasonable to take the drag proportional to the square of the mean velocity, U^2. In this case the formula of Lions[5,13] is recovered:

$$\mu_T(\nabla u + \nabla u^T)n - pn = |[u]|M[u] \quad \text{on } \Sigma,$$

where M is a second-order tensor.

With this boundary condition the variational formulation in $\Omega - \Sigma$, where Σ is the slab assimilated to a line, is

$$\int_{\Omega-\Sigma} (\partial_t u + u\nabla u)v + \mu_T(\nabla u + \nabla u^T) : (\nabla v + \nabla u^T) - \int_\Sigma F(c, [u])[v] = 0$$

for all $v \in J(\Omega - \Sigma)$.

4. NUMERICAL TABULATION FOR A WAVY SURFACE

Carreau[1] and Morisset[6] tabulated the cell problem for a compressible flow at high Mach number. We reproduced their simulations at low Mach number with another code. The results are summarized in Table I.

The geometry and flow visualization are shown later in Figure 8.

Table I. Stress tensor versus Reynolds number, showing independence of mean stress w.r.t. height

		Wall	$y = 0{\cdot}01$	$y = 0{\cdot}05$	$y = 0{\cdot}1$
$Re = 1 \times 10^4$	Tangential stress	−0·003179	−0·003512	−0·003115	−0·002876
	Normal stress	7·75155	7·93776	7·93790	7·93794
$Re = 5 \times 10^4$	Tangential stress	−0·004627	−0·004872	−0·003931	−0·002728
	Normal stress	7·75261	7·93822	7·93893	7·9390
$Re = 1 \times 10^5$	Tangential stress	−0·004858	−0·005122	−0·004326	−0·002744
	Normal stress	7·75147	7·93654	7·93776	7·93802
$Re = 1 \times 10^6$	Tangential stress	−0·003845	−0·003876	−0·004079	−0·003008
	Normal stress	7·75035	7·93501	7·93634	7·93653

4.1. Test on a flat plate for laminar flow

The previous analysis should work even in the limit of a flat plate whose irregularities tend to zero. Then the periodic cell becomes a vertical line and the computational domain becomes a half-plane above the flat plate. Thus the cell problem is obtained by dropping all tangential derivatives in the Navier–Stokes equations:

$$-\nu\partial_{\mathrm{n}}^2 u + \partial_{\mathrm{s}} p = 0.$$

The solution is a parabolic profile when $\partial_{\mathrm{s}} p$ is constant:

$$u = \frac{\partial_{\mathrm{s}} p}{2\nu} y^2 + \frac{y}{\delta}\left(-\frac{\partial_{\mathrm{s}} p}{2\nu}\delta^2 + u|_{y=\delta}\right).$$

The relation between the normal derivative and itself is easy to find:

$$\nu\partial_{\mathrm{n}} u + u\frac{\nu}{\delta} + \frac{\partial_{\mathrm{s}} p}{2}\delta = 0.$$

This boundary condition has been tested on the flow over a flat plate for two values of δ, $\delta = 0{\cdot}01$ and $0{\cdot}1$, with $\nu = 0{\cdot}003$ ($\delta^+ = \delta\surd[(\partial u/\partial y)\nu^{-1}] = 0{\cdot}01\ \surd(10^6 \times 0{\cdot}2/9) = \delta \times\ 0{\cdot}1414 \times 10^3/3 = 50\delta$) (Figures 4–7). The numerical results show that it works for $y^+ < 0{\cdot}5$, which is much less than the values used for turbulent boundary layers at $Re = 300$ (i.e. $\nu = 1/300$, $h = 1$, $u_\infty = 1$, where h is the height of the computational domain).

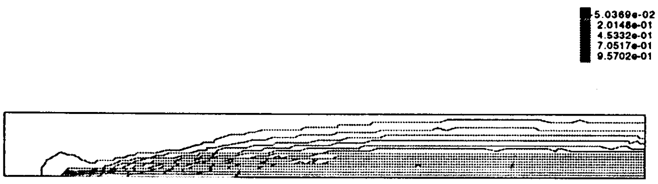

Figure 4. Navier–Stokes solution with no slip ($u = 0$) on flat plate

Figure 5. Navier–Stokes solution with a laminar wall law and $\delta = 0{\cdot}1$

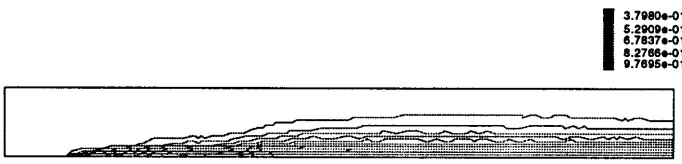

Figure 6. Navier–Stokes solution with a laminar wall law and $\delta = 0{\cdot}1$

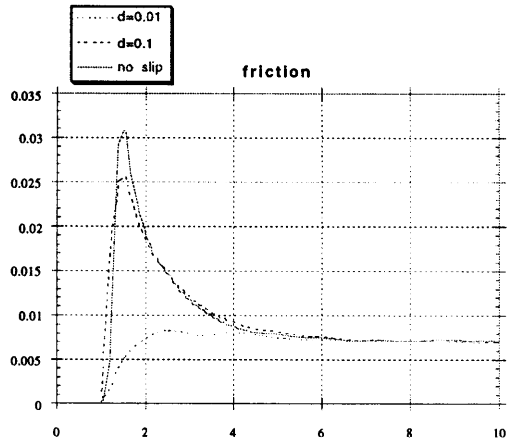

Figure 7. Friction at wall as a function of x

This small example also shows the limit of this wall law approach: it is a viscous matching and has nothing to do with Prandtl's boundary layer analysis.

5. WALL LAWS AND LOW-*Re* CORRECTIONS

5.1. Smooth surface

Let us apply the same idea to the k–ε model with low-Reynolds-number corrections:

$$\mu_{\mathrm{T}} = c_\mu \rho \frac{k^2}{\varepsilon}, \quad \text{with } D_t = \partial_t + u\Delta,$$

$$E = \tfrac{1}{2}|\nabla u + \nabla u^{\mathrm{T}}|^2 - \tfrac{1}{2}|\nabla \cdot u|^2,$$

$$D_t k - \frac{\sigma_k}{\rho}\nabla\cdot(\mu_{\mathrm{T}}\nabla k) + k\left(\frac{\varepsilon}{k} + \tfrac{2}{3}\nabla\cdot u\right) = c_\mu \frac{k^2}{\varepsilon} E,$$

$$D_t \varepsilon - \frac{\sigma_\varepsilon}{\rho}\nabla\cdot(\mu_{\mathrm{T}}\nabla \varepsilon) + \varepsilon\left(c_2\frac{\varepsilon}{k} + \frac{2c_1}{3c_\mu}\nabla\cdot u\right) = c_1 k E.$$

The low-Reynolds-number corrections are

$$c'_\mu = f_\mu c_\mu, \qquad c'_1 = f_1 c_1, \qquad c'_2 = f_2 c_2,$$

$$f_\mu = 1(-\mathrm{e}^{-0\cdot 017 y\nu^{-1}\sqrt{k}})\left(1 + 20\cdot 5\frac{\nu\varepsilon}{k^2}\right),$$

$$f_1 = 1 + \left(\frac{0\cdot 05}{f_\mu}\right)^3, \qquad f_2 = 1 - \mathrm{e}^{\nu\varepsilon k^{-2}}.$$

We are in fact in the same situation as for the laminar flat plate: two scales, one due to the strong gradients in the normal direction and the other associated with the other gradients. Domain

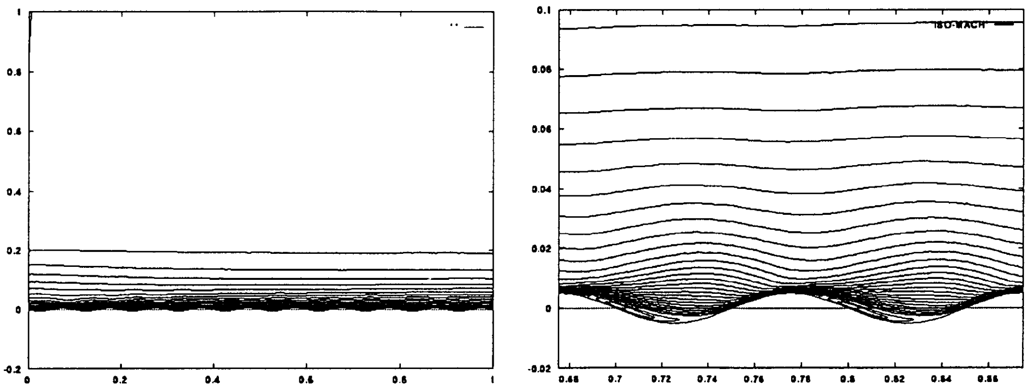

Figure 8. Mach lines (left) and zoom of centre part (right) for flow over a rough boundary, used to compute Table I

decomposition will give a boundary condition relating the velocity and its gradient on a border at a small distance from the physical boundary.

As in the flat plate case, there are no lateral oscillations, so the cell problem is on a vertical line, i.e. all tangential derivatives are dropped. In the stationary case an analytical solution is found; it is the wall law when $5 \leq yu^*/v) \leq 50$:

$$\frac{u}{u^*} = \frac{1}{\chi}\log\left(\frac{yu^*}{v}\right) + \beta + y^+ \frac{v}{\chi u^{*2}}\frac{\partial p}{\partial s}.$$

Next eliminate u^* by differentiating the log law:

$$\frac{1}{u^*}\frac{\partial u}{\partial y} = \frac{1}{y\chi} + \frac{1}{\chi u^*}\frac{\partial p}{\partial s},$$

giving

$$u\cdot s = y\left(\chi\frac{\partial u\cdot s}{\partial n} - \frac{\partial p}{\partial s}\right)\left\{\log\left[\frac{y^2}{v}\left(\chi\frac{\partial u\cdot s}{\partial n} - \frac{\partial p}{\partial s}\right)\right] + \beta\right\},$$

which, written at $y = \delta$, gives the required boundary condition.

Usually $\partial p/\partial s$ is dropped because it is small compared with $\partial u/\partial n$, but in Reference 7 it is shown that this terms helps capture recirculations numerically.

5.2. *Turbulent flow over a wavy surface*

When the drag of a wavy surface is assumed proportional to u^2, the domain decomposition approach's answer to the same problem is

$$v_{\mathrm{T}}\frac{\partial u}{\partial n} = c(v_{\mathrm{T}})u|u|.$$

However, the wall law being valid at the matching interface, this boundary condition could also be used with u given by the law of the wall. It gives

$$u^{*2} = c(v_{\mathrm{T}})u^2 = c(v_{\mathrm{T}})u^{*2}\left(\frac{1}{\chi}\log\,\delta^+ + \beta\right)^2,$$

i.e.

$$\beta = c(\nu_{\mathrm{T}})^{-1/2} - \frac{1}{\chi}\log \delta^{+}.$$

Thus the effect of the roughness is to change the value of $c(\nu_{\mathrm{T}})$ and hence to shift the value of β by $c_{\mathrm{wavy}}(\nu_{\mathrm{T}})^{-1/2} - c_{\mathrm{flat}}(\nu_{\mathrm{T}})^{-1/2}$.

6. ERROR ESTIMATES

The main result[8] compares the exact solution u^{ε} with the solution u^0 above a mean surface Σ with a Frechet boundary condition

$$\|u^{\varepsilon} - u^0\|_{\Omega^0} C(\varepsilon\|\partial_s\chi\|_{0,S} + \varepsilon^{3/2}),$$

where χ is the solution of the cell problem which defines the constant in the Frechet boundary condition. This result shows that the smooth artificial boundary Σ should be sufficiently far from the wavy boundary so as to have $\|\partial_{s\chi}\|_{0,S} = O(\varepsilon^{1/2})$, which is possible because χ tends to a function independent of s at infinity.

6.1. Stokes Flow

For Stokes flow the mean flow away from the rough surface is found by

$$-\nu\Delta u^0 + \Delta p^0 = 0, \qquad \nabla\cdot u^0 = 0 \quad \text{in } \Omega,$$

$$-\nu\langle\chi\rangle\partial_n u^0 + p^0 n + \frac{1}{\varepsilon}u^0 = 0 \quad |\text{on } \Sigma, \qquad u^0|_{\Gamma_1} = g,$$

where the matrix $\chi = \{\chi^1, \chi^2, \chi^3\}$ has χ^i the solution of

$$-\nu\Delta\chi + \nabla\eta = 0, \qquad \nabla\cdot\chi = 0,$$

with periodic conditions on the lateral boundaries, $\chi = 0$ on the lower boundary and on the upper boundary S of the cell domain

$$-\nu\partial_n\chi^i + \eta^i n = E^i, \quad \text{with } E^i_j = \delta_{ij}.$$

Because $\nabla\cdot\chi = 0$, we have that $\langle\chi\rangle|_S = 0$, so that u^0·n=0 on Σ.

For Navier–Stokes equations there is an interaction with the boundary layer which acts at the next level. Thus at order one the decomposition is the same, but at order $\varepsilon^{3/2}$ the boundary condition is non-linear, as will be shown in a future publication.

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